Solution of Triangle

Solution of triangle refers to values of sides and angles of the triangle. In triangle, there are three angles and three sides. If we specify any three values of them, we may or may not be able to completely determine the remaining sides/angles. For example, we three sides are given (SSS), then the triangle is unique and we can determine all remaining angles. However, if three angles are given (AAA), then it is not possible to determine the sides from it. In this chapter, we denote the opposite sides of triangle ABC by a, b, and c. While solving for triangles, we should be careful of following facts

(i) The sum of any two sides must be greater than the third side. So we cannot construct a triangle with sides 1, 2 and 5.

(ii) From **Sine Law**, the sides are proportional to the ratio of sine of angles. Therefore, we except our side to be greater than other if it's opposite angle is greater than other. Also only one obtuse angle is possible on the triangle.

Our technique for determining the sides and angle are **Cosine Law**

$$a^2 = b^2 + c^2 - 2bc\cos A \tag{1}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \tag{2}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{3}$$

and **Sine Law**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{4}$$

where $2R$ is diameter of circumcirle. Equation (1) can be used to solve for side a or angle A and similarly for others. Now, we examine the following cases.

1. If three sides are given (SSS).

   If three sides are given then the only parameters that are unknown are the three angles of triangle. They can be found using **Cosine Law**. **For example**, if $a = 3$, $b = 4$ and $c = 5$ then

$$\begin{aligned}\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{5^2 + 4^2 - 3^2}{2 \cdot 5 \cdot 4} \\ &= \frac{4}{5}\end{aligned}$$

This gives $\cos A = \frac{4}{5} \implies A = \cos^{-1}(\frac{4}{5}) \approx 36.87°$. Similarly, we can find the angle B and angle C can be found by $C = 180° - A - B$.

2. If two angles and a side is given (AAS/ASA).

   If two angles are given then the third angle is also known (as sum of angles is 180°) so ASA and AAS are equivalent. Since we know one side and all three angles are known, we can find sides by **Sine Law**. **For example**, suppose $a = 3$, $B = 30°$, $C = 45°$ and we are supposed to find measure of all sides. First of all $A = 180° - B - C = 105°$. Then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

   Then

$$b = \sin B \cdot \frac{a}{\sin A} = \sin 30° \cdot \frac{3}{\sin 105°} \approx 1.55$$

   and

$$c = \sin C \cdot \frac{a}{\sin A} = \sin 45° \cdot \frac{3}{\sin 105°} \approx 2.19$$

3. If two sides are given and one angle is given (SAS/SSA).

   In this case when angle is in between two sides (SAS), we can find the third side using Cosine Law. The other angle can be found using the Cosine Law. **For example**, if $a = 3$, $C = 30°$ and $b = 4$ then

$$c^2 = a^2 + b^2 - 2ab\cos C = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos 30° \approx 4.22$$

   This gives $c \approx \sqrt{4.22} = 2.05$. Similarly,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{3^2 + 2.05^2 - 4^2}{2 \cdot 3 \cdot 2.05} \approx -0.226$$

   This gives $B = \cos^{-1}(-0.226) = 103.06°$ and $A = 180° - B - C$.

   In the case that angle is not between two sides (SSA), there may exist **two solution**, a **unique solution** or **no solution** at all. This is because we need to find the value of angle using Sine law. Suppose a, b and B are given then angle A can be determined by Sine law as

$$\frac{\sin A}{a} = \frac{\sin B}{b} \implies \sin A = \frac{a}{b}\sin B$$

   (a) Now if $0 < \frac{a}{b}\sin B < 1$ then there are two values of A as the other value is $180° - A$. If A is acute then $180° - A$ is obtuse.

      i. If $a < b$ then $A < B$. Since both angles A and B cannot be obtuse, A must be acute. This case there is only one solution.

      ii. If $a = b$ then the triangle is isosceles so $A = B$.

      iii. if $a > b$ then both both A and $180° - A$ are solutions. This case is called **ambiguous** case.

(b) If $\frac{a}{b}\sin B = 1$ then $\sin A = 1 \implies A = 90^\circ$ so there is unique solution. This happens to be precisely RHS case.

(c) If $\frac{a}{b}\sin B > 1$ then $\sin A > 1$. Since the value of sin cannot be greater than one, there is no **solution**. For exercise, solve the triangle in each of following cases.

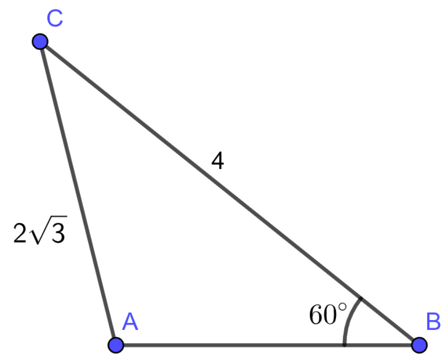
i. $a = 4, b = 2, \angle B = 60^\circ$

ii. $a = 4, b = 2\sqrt{3}, \angle B = 60^\circ$

iii. $a = 4, b = 4, \angle B = 60^\circ$

iv. $a = 4, b = 5, \angle B = 60^\circ$

Before solving, try drawing the image as well.



Finally, avoid **Sine Law** to get the value of **angle** as much as possible. If you are using it for values of angles then make sure to look for sub-case of (a) of SSA.